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SOME ASPECTS OF THE EXACT FOLDY-WOUTHUYSEN TRANSFORMATION FOR A DIRAC FERMION

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The Foldy-Wouthuysen transformation (FWT) is used to separate distinct components of relativistic spinor field, e.g. electron and positron. Usually, the FWT is perturbative, but in some cases there is an involution operator and the transformation can be done exactly. We consider some aspects of an exact FWT and show that, even if the theory does not admit an involution operator, one can use the technique of exact FWT to obtain the conventional perturbative result. Several particular cases can be elaborated as examples.

Keywords: Electromagnetic Field; Dirac Particle; Foldy-Wouthuysen transformation.

1. Introduction

The Foldy-Wouthuysen transformation (FWT)¹ is a useful method to extract physical information from Dirac equation in presence of external fields. For example, take the Dirac equation in external electromagnetic field and suppose that ψ has the bi-spinor form

$$i\hbar\frac{\partial}{\partial t}\psi = [c\vec{\alpha} \cdot (\vec{p} - \frac{e}{c}\vec{A}) + \beta mc^2 + eA_0]\psi \quad \text{where} \quad \psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{\frac{-imc^2 t}{\hbar}}. \quad (1)$$

Substituting the second equation into the first one we get two equations that represent a mixture of the fields φ and χ . In order to separate the equations for φ and χ , one can perform the well known Foldy-Wouthuysen transformation². The first step is to define even and odd operators. An odd operator anticommutes with β , it is mixing the two components of the wave function. Even operators commute with β and do not produce mixing.

In order to perform the FWT, the Hamiltonian is written in the form $H = \beta m + \varepsilon + \vartheta$, where ε are even operators and ϑ are odd ones. The usual perturbative FWT is an expansion in powers of $(1/m)$. The transformed (purely even) Hamiltonian in the order $(1/m^3)$ has the form (see, e.g. Ref. 2)

$$H''' = \beta \left(m + \frac{\vartheta}{2m} + \frac{\vartheta^2}{8m^3} \right) + \varepsilon + \frac{1}{m^2}[\vartheta, [\vartheta, \varepsilon]] - \frac{1}{8m^2}[\vartheta, \dot{\vartheta}]. \quad (2)$$

2 Bruno Gonçalves

As it was already mentioned, this result is perturbative. On the other hand, in some cases there is the possibility to obtain an exact solution with separated components of a Dirac field. In order to perform the EFWT, it is necessary to check whether the Hamiltonian of the theory falls into the class of models which admit the anticommuting involution operator $J = i\gamma_5\beta$. There are many papers describing the algorithm to perform this transformation, e.g. Refs. 3, 4, 5, 6.

The operator J is Hermitian, $J^\dagger = J$, and unitary, $JJ^\dagger = J^2 = 1$ at the same time. Moreover, it anticommutes both with the Hamiltonian $J\hat{\mathcal{H}} + \hat{\mathcal{H}}J = 0$ and with the β matrix $J\beta + \beta J = 0$. If this condition is satisfied the transformed Hamiltonian has the form

$$H^{tr} = UHU^* = \beta[\sqrt{H^2}]^{EVEN} + J[\sqrt{H^2}]^{ODD}. \quad (3)$$

As a result of EFWT one obtains H^2 and, afterwards, it is possible to expand the square root of the operator H^2 in some parameter considered small in the theory. For example, if one chooses the parameter $1/m$, the resulting Hamiltonian describes the theory in the non-relativistic limit. Let us note that the result of this procedure can, in principle differ from the one of the perturbative FWT⁴. Once the transformed Hamiltonian is obtained, in order to make some interpretations of the result, the next step is to present the Dirac fermion ψ in the bi-spinor form (1) and use the equation $i\hbar\partial_t\psi = H\psi$ to derive the Hamiltonian for the field φ .

2. A method for the EFWT

If the theory has many external fields, the operator H^2 in equation (3) can becomes cumbersome. In order to avoid this situation, one can use some notations which can be useful in many different theories. Let us consider a Hamiltonian in general form

$$H = \beta mc^2 + \beta q + \alpha^j K_j^i \partial_i + \alpha^i g_i, \quad \text{where} \quad K_j^i = -i\hbar c(\delta_j^i + T_j^i v). \quad (4)$$

In these notations, the matrix T_j^i has numerical parameters and q is a constant. The external fields are introduced by the scalar function v and the vector g_i . According to the standard prescription⁵, the first step in deriving the EFWT is to calculate the square of the Hamiltonian H^2 . A direct calculus gives, after certain algebra, the following result for the square of the Hamiltonian H^2

$$\begin{aligned} H^2 = & m^2 c^4 + K^{il} K_l^m \partial_i \partial_m + 2 K_j^i g^j \partial_i + g^2 + K^{il} \partial_i (K_l^m) \partial_m + K_j^i \partial_i (g^j) \\ & + i \Sigma_k \epsilon^{jlk} K_j^i \partial_i (K_l^m) \partial_m + i \Sigma_k \epsilon^{jlk} K_j^i \partial_i (g_l) + 2mc^2 q + \alpha^j K_j^i \partial_i (q) \beta. \end{aligned} \quad (5)$$

Let us notice that the components of the matrix K_j^i depend exclusively on v while the components of the vector g_i can also depends on other external fields. The expression (5) is rather general, but it can also be formulated in case that T_j^i and g_i depend on α matrices¹. Moreover, this result includes various other particular known cases. The main point is to choose the correct interaction term g_i . In order to illustrate this fact, if we put $v = 0$, the matrix K_j^i becomes $-i\hbar c\delta_j^i$. Then, with the term g_i equal to zero, we have the free particle case. In case $g_i = -eA_i$, we have

the particle is in presence of magnetic field and in case $g_i = -eA_i + \alpha_i \mu_I \eta \vec{\Sigma} \cdot \vec{B}$, we meet the particle with the anomalous magnetic moment in the magnetostatic field \vec{B} . All these cases were tested and gave the same well-known results from the literature⁵.

3. Semi-exact transformation

Let us now study the case when the Hamiltonian does not anticommute with the involution operator. The EFWT is not allowed anymore, but it is possible to obtain an even Hamiltonian using equation (3). The method can be explained using a simple example. Let us suppose a Dirac particle interacting with constant and uniform magnetic and scalar electromagnetic potential fields. The Hamiltonian reads

$$H = c\vec{\alpha} \cdot \vec{p} - e\vec{\alpha} \cdot \vec{A} + e\Phi + \beta mc^2. \quad (6)$$

Direct inspection shows that the term $e\Phi$ does not anticommute with the involution operator. Thus, the Hamiltonian (6) does not enable one to perform the EFWT.

Let us make a modification of the term $e\Phi$, that is multiply it by the β -matrix. The modified term anticommutes with the involution operator and now the exact transformation is perfectly possible. The main point is that, in the linear order in $1/m$, an extra factor of β has no effect. The reason is that, after deriving the final Hamiltonian operator, it will have the block diagonal structure. We are interested only in the upper block of Hamiltonian which is even (after transformation) to perform physical analysis. Therefore, it does not matter if this term is multiplied by β or not, because beta has a two block form with the unity matrix (in standard representation) in the relevant block. As a result we arrive at what one can call semi-exact Foldy-Wouthuysen transformation, because it is exact in only part of external fields and linear in other external fields. After all, the Hamiltonian we are going to deal with has the form

$$H = c\vec{\alpha} \cdot \vec{p} - e\vec{\alpha} \cdot \vec{A} + e\beta\Phi + \beta mc^2. \quad (7)$$

Following all the steps described in the introduction and using the result (5) to the equation (7), the non-relativistic transformed Hamiltonian is

$$H^{tr} = \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 + e\Phi + \frac{\hbar e}{2mc} \vec{B}. \quad (8)$$

This result is in perfect agreement with the one obtained using the perturbative approach in Ref. 7 if the torsion fields are equal to zero. The advantage here is that the result can be obtained in a much more economic way. The semi-exact Foldy-Wouthuysen transformation described above can also be applied to a more complex Hamiltonian such as Dirac particle interacting with constant magnetic and torsion field which were previously treated perturbatively in Refs. 8, 9. The preliminary report on the result of the semi-exact calculation will be published in Ref. 10.

4 *Bruno Gonçalves*

4. Conclusions

In this paper, the relation between the two different methods used to obtain an even Dirac Hamiltonian were discussed. They were illustrated with simple examples, but can also be applied to more complex Hamiltonians. In section (2), some notations were introduced and with them it was possible to write a very general solution that can be applied to many theories, for example Ref. 1. In section (3) there is a formulation, using an explicit example, of what was called the semi-exact Foldy Wouthuysen transformation, that is a transformation that makes the Hamiltonian even with a method of calculation very similar to the one used in the EFWT, for a Hamiltonian that does not admit the involution operator.

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References

1. B. Gonçalves, Yu.N. Obukhov and I.L. Shapiro, *Phys. Rev. D* **75** (2007) 124023.
2. J.M. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill Book Company, NY, 1964).
3. C.G. de Oliveira and J. Tiomno, *Nuovo Cim.* **24** (1962) 672.
4. Yu.N. Obukhov, *Phys. Rev. Lett.* **86** (2001) 192.
5. E. Eriksen and M. Kolsrud, *Nuovo Cim. Suppl.* **18** (1960) 1.
6. A.G. Nikitin, *J. Phys. A: Math. Gen.* **A31** (1998) 3297.
7. V.G. Bagrov, I.L. Buchbinder and I.L. Shapiro, *Sov. J. Phys.* **35** (1992) 5; hep-th/9406122.
8. I.L. Shapiro, *Phys. Repts.* **357** (2002) 113.
9. L.H. Ryder and I.L. Shapiro, *Phys. Lett.* **A247** (1998) 21.
10. B. Gonçalves, Yu.N. Obukhov and I.L. Shapiro *Using Exact Foldy-Wouthuysen for a Dirac Fermion in Torsion and Magnetic Fields Background*. To be published in the volume dedicated to 60 years of Professor I. L. Buchbinder.